

TOPOLOGICAL AND DIFFERENTIABLE RIGIDITY OF SUBMANIFOLDS IN SPACE FORMS*

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Abstract

Let $F^{n+p}(c)$ be an $(n+p)$ -dimensional simply connected space form with nonnegative constant curvature c . We prove that if $M^n (n \geq 4)$ is a compact submanifold in $F^{n+p}(c)$, and if $Ric_M > (n-2)(c + H^2)$, where H is the mean curvature of M , then M is homeomorphic to a sphere. We also show that the pinching condition above is sharp. Moreover, we obtain a new differentiable sphere theorem for submanifolds with positive Ricci curvature.

1 Introduction

The investigation of curvature and topology of Riemannian manifolds and submanifolds is one of the main stream in global differential geometry. In 1951, Rauch first proved a topological sphere theorem for positive pinched compact manifolds. During the past sixty years, there are many progresses on sphere theorems for Riemannian manifolds and submanifolds [2, 5, 8, 12, 13, 19]. Recently Böhm and Wilking [3] proved that every manifold with 2-positive curvature operator must be diffeomorphic to a space form. More recently, Brendle and Schoen [6] proved the remarkable differentiable sphere theorem for manifolds with pointwise $1/4$ -pinched curvatures. Moreover, Brendle and Schoen [7] obtained a differentiable rigidity theorem for compact manifolds with weakly $1/4$ -pinched curvatures in the pointwise sense. The following important convergence result for the normalized Ricci flow in higher dimensions, initiated by Brendle and Schoen [6], was finally verified by Brendle [4].

Theorem A. *Let (M, g_0) be a compact Riemannian manifold of dimension $n(\geq 4)$. Assume that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)} g(t)$$

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exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

Let M^n be an $n(\geq 2)$ -dimensional submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . Denote by H and S the mean curvature and the squared length of the second fundamental form of M , respectively. After the pioneering rigidity theorem for minimal submanifolds in a sphere due to Simons [21], Lawson [15] and Chern-do Carmo-Kobayashi [9], Yau [31] and Ejiri [10] obtained three important rigidity theorems for oriented compact minimal submanifolds in S^{n+p} . In 1990, the first named author [25] proved the following generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

Theorem B. *Let M be an n -dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$ -dimensional unit sphere S^{n+p} . If $S \leq C(n, p, H)$, then M is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an $(n+1)$ -sphere, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C(n, p, H)$ is defined by*

$$C(n, p, H) = \begin{cases} \alpha(n, H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \frac{n}{2-\frac{1}{p}}, & \text{for } p \geq 2 \text{ and } H = 0, \\ \min \left\{ \alpha(n, H), \frac{n+nH^2}{2-\frac{1}{p-1}} + nH^2 \right\}, & \text{for } p \geq 3 \text{ and } H \neq 0, \end{cases}$$

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Later, the above pinching constant $C(n, p, H)$ was improved, by Li-Li [17] for $H = 0$ and by Xu [26] for $H \neq 0$, to

$$C'(n, p, H) = \begin{cases} \alpha(n, H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \min \left\{ \alpha(n, H), \frac{1}{3}(2n + 5nH^2) \right\}, & \text{otherwise.} \end{cases}$$

Using nonexistence for stable currents on compact submanifolds of a sphere and the generalized Poincaré conjecture in dimension $n(\geq 5)$ verified by Smale, Lawson and Simons [16] proved that if $M^n(n \geq 5)$ is an oriented compact submanifold in S^{n+p} , and if $S < 2\sqrt{n-1}$, then M is homeomorphic to a sphere. Let $F^{n+p}(c)$ be an $(n+p)$ -dimensional simply connected space form with nonnegative constant curvature c . Putting

$$\alpha(n, H, c) = nc + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)cH^2},$$

we have $\min_H \alpha(n, H, c) = 2\sqrt{n-1}c$. Motivated by the rigidity theorem above, Shiohama and Xu [20] improved Lawson-Simons' result and proved the optimal sphere theorem.

Theorem C. *Let $M^n(n \geq 4)$ be an oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. Suppose that $\sup_M (S - \alpha(n, H, c)) < 0$. Then M is homeomorphic to a sphere.*

The study of differentiable pinching problem for submanifolds was initiated by Xu and Zhao [30]. Making use of the convergence results of Hamilton and Brendle for Ricci flow

and the Lawson-Simons formula for the nonexistence of stable currents, Gu and Xu [11, 27] proved the following differentiable sphere theorem for submanifolds in space forms.

Theorem D. *Let M be an $n(\geq 4)$ -dimensional oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. Assume that $S \leq 2c + \frac{n^2 H^2}{n-1}$, where $c + H^2 > 0$. We have*

- (i) *If $c = 0$, then M is either diffeomorphic to S^n , \mathbb{R}^n , or locally isometric to $S^{n-1}(r) \times \mathbb{R}$.*
- (ii) *If M is compact, then M is diffeomorphic to S^n .*

When M is compact and $c = 0$, Andrews-Baker [1] obtained the same sphere theorem by the convergence result for mean curvature flow independently.

Recently, Xu and Gu [28] proved the following generalized Ejiri rigidity theorem for compact submanifolds with parallel mean curvature in space forms.

Theorem E. *Let M be an $n(\geq 3)$ -dimensional oriented compact submanifold with parallel mean curvature in $F^{n+p}(c)$ with $c + H^2 > 0$. If*

$$Ric_M \geq (n-2)(c + H^2),$$

then M is either a totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, a Clifford hypersurface $S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}})$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3}(c + H^2))$ in $S^7(\frac{1}{\sqrt{c+H^2}})$. Here $\mathbb{C}P^2(\frac{4}{3}(c + H^2))$ denotes the 2-dimensional complex projective space minimally immersed in $S^7(\frac{1}{\sqrt{c+H^2}})$ with constant holomorphic sectional curvature $\frac{4}{3}(c + H^2)$.

The purposes of the present paper is to investigate rigidity of topological and differentiable structures of compact submanifolds. Our paper is organized as follows. Some notations and lemmas are prepared in Section 2. In Section 3, we prove that if M is an $n(\geq 4)$ -dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$, and if $Ric_M > (n-2)(c + H^2)$, then M is homeomorphic to a sphere. We then give an example to show that the pinching condition above is sharp. Moreover, we obtain a new differentiable sphere theorem for compact submanifolds with positive Ricci curvature in a space form.

2 Notations and lemmas

Throughout this paper let M^n be an n -dimensional compact submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+p, \quad 1 \leq i, j, k, \dots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \dots \leq n+p.$$

For an arbitrary fixed point $x \in M \subset N$, we choose an orthonormal local frame field $\{e_A\}$ in N^{n+p} such that e_i 's are tangent to M . Denote by $\{\omega_A\}$ the dual frame field of $\{e_A\}$. Let Rm , h and ξ be the Riemannian curvature tensor, second fundamental form and mean

curvature vector of M respectively, and \overline{Rm} the Riemannian curvature tensor of N . Then

$$\begin{aligned} Rm &= \sum_{i,j,k,l} R_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l, \\ \overline{Rm} &= \sum_{A,B,C,D} \overline{R}_{ABCD} \omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D, \\ h &= \sum_{\alpha,i,j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha,i} h_{ii}^\alpha e_\alpha, \\ R_{ijkl} &= \overline{R}_{ijkl} + \sum_{\alpha} (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \end{aligned} \tag{2.1}$$

$$R_{\alpha\beta kl} = \overline{R}_{\alpha\beta kl} + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta). \tag{2.2}$$

We define

$$S = |h|^2, \quad H = |\xi|, \quad H_\alpha = (h_{ij}^\alpha)_{n \times n}.$$

Denote by $Ric(u)$ the Ricci curvature of M in direction of $u \in UM$. From the Gauss equation, we have

$$Ric(e_i) = \sum_j \overline{R}_{ijij} + \sum_{\alpha,j} [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2]. \tag{2.3}$$

Set $Ric_{\min}(x) = \min_{u \in U_x M} Ric(u)$. Denote by $K(\pi)$ the sectional curvature of M for tangent 2-plane $\pi(\subset T_x M)$ at point $x \in M$, $\overline{K}(\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_x N)$ at point $x \in N$. Set $\overline{K}_{\min} := \min_{\pi \subset T_x N} \overline{K}(\pi)$, $\overline{K}_{\max} := \max_{\pi \subset T_x N} \overline{K}(\pi)$. Then by Berger's inequality, we have

$$|\overline{R}_{ABCD}| \leq \frac{2}{3} (\overline{K}_{\max} - \overline{K}_{\min}) \tag{2.4}$$

for all distinct indices A, B, C, D .

When the ambient manifold N^{n+p} is the complete and simply connected space form $F^{n+p}(c)$ with constant curvature c , we have

$$\overline{R}_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.5}$$

Then the scalar curvature of M is given by

$$R = n(n-1)c + n^2 H^2 - S. \tag{2.6}$$

The nonexistence theorem for stable currents in a compact Riemannian manifold M isometrically immersed into $F^{n+p}(c)$ is employed to eliminate the homology groups $H_q(M; \mathbb{Z})$ for $0 < q < n$, which was initiated by Lawson-Simons [16] and extended by Xin [24].

Theorem 2.1. *Let M^n be a compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. Assume that*

$$\sum_{k=q+1}^n \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] < q(n-q)c$$

holds for any orthonormal basis $\{e_i\}$ of $T_x M$ at any point $x \in M$, where q is an integer satisfying $0 < q < n$. Then there does not exist any stable q -currents. Moreover,

$H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0$, and $\pi_1(M) = 0$ when $q = 1$. Here $H_i(M; \mathbb{Z})$ is the i -th homology group of M with integer coefficients.

To prove the sphere theorems for submanifolds, we need to eliminate the fundamental group $\pi_1(M)$ under the Ricci curvature pinching condition, and get the following lemma.

Lemma 2.1. *Let M be an $n(\geq 4)$ -dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. If the Ricci curvature of M satisfies*

$$Ric_M > \frac{n(n-1)}{n+2}(c + H^2),$$

then $H_1(M; \mathbb{Z}) = H_{n-1}(M; \mathbb{Z}) = 0$, and $\pi_1(M) = 0$.

Proof. From (2.6) and the assumption, we have

$$S - nH^2 < \frac{2n(n-1)}{n+2}(c + H^2).$$

This together with (2.3) implies that

$$\begin{aligned} & \sum_{k=2}^n [2|h(e_1, e_k)|^2 - \langle h(e_1, e_1), h(e_k, e_k) \rangle] \\ &= 2 \sum_{\alpha} \sum_{k=2}^n (h_{1k}^{\alpha})^2 - \sum_{\alpha} \sum_{k=2}^n h_{11}^{\alpha} h_{kk}^{\alpha} \\ &= \sum_{\alpha} \sum_{k=2}^n (h_{1k}^{\alpha})^2 - Ric(e_1) + (n-1)c \\ &\leq \frac{1}{2}(S - nH^2) - Ric(e_1) + (n-1)c \\ &< \frac{n(n-1)}{n+2}(c + H^2) - \frac{n(n-1)}{n+2}(c + H^2) + (n-1)c \\ &= (n-1)c. \end{aligned} \tag{2.7}$$

This together with Theorem 2.1 implies that $H_1(M; \mathbb{Z}) = H_{n-1}(M; \mathbb{Z}) = 0$, and $\pi_1(M) = 0$. This proves Lemma 2.1.

3 Sphere theorems for submanifolds

In this section, we investigate rigidity of topological and differentiable structures of compact submanifolds in space forms. Motivated by Theorem E, we first prove the following topological sphere theorem for compact submanifolds in space forms.

Theorem 3.1. *Let M be an $n(\geq 4)$ -dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. If*

$$Ric_M > (n-2)(c + H^2),$$

then M is homeomorphic to a sphere.

Proof. Assume that $2 \leq q \leq \frac{n}{2}$. Setting

$$T_\alpha := \frac{\text{tr} H_\alpha}{n},$$

we have $\sum_\alpha T_\alpha^2 = H^2$, and

$$\text{Ric}(e_i) = (n-1)c + \sum_\alpha \left[nT_\alpha h_{ii}^\alpha - (h_{ii}^\alpha)^2 - \sum_{j \neq i} (h_{ij}^\alpha)^2 \right]. \quad (3.1)$$

Then we get

$$\begin{aligned} & \sum_{k=q+1}^n \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] \\ &= 2 \sum_\alpha \sum_{k=q+1}^n \sum_{i=1}^q (h_{ik}^\alpha)^2 - \sum_\alpha \sum_{k=q+1}^n \sum_{i=1}^q h_{ii}^\alpha h_{kk}^\alpha \\ &= \sum_\alpha \left[2 \sum_{k=q+1}^n \sum_{i=1}^q (h_{ik}^\alpha)^2 - \left(\sum_{i=1}^q h_{ii}^\alpha \right) \left(\text{tr} H_\alpha - \sum_{i=1}^q h_{ii}^\alpha \right) \right] \\ &\leq \sum_\alpha \left[2 \sum_{k=q+1}^n \sum_{i=1}^q (h_{ik}^\alpha)^2 - nT_\alpha \sum_{i=1}^q h_{ii}^\alpha + q \sum_{i=1}^q (h_{ii}^\alpha)^2 \right] \\ &\leq q \sum_{i=1}^q [(n-1)c - \text{Ric}(e_i)] + n(q-1) \sum_\alpha \sum_{i=1}^q T_\alpha h_{ii}^\alpha \\ &\leq q^2[(n-1)(c + H^2) - \text{Ric}_{\min}] \\ &\quad - q(n-q)H^2 + n(q-1) \sum_\alpha \sum_{i=1}^q T_\alpha (h_{ii}^\alpha - T_\alpha) \\ &\leq q(n-q)[(n-1)(c + H^2) - \text{Ric}_{\min}] \\ &\quad - q(n-q)H^2 + n(q-1) \sum_\alpha \sum_{i=1}^q T_\alpha (h_{ii}^\alpha - T_\alpha). \end{aligned} \quad (3.2)$$

On the other hand, we obtain

$$\begin{aligned} & \sum_{k=q+1}^n \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] \\ &= \sum_\alpha \left[2 \sum_{k=q+1}^n \sum_{i=1}^q (h_{ik}^\alpha)^2 - \frac{n-q}{n} \left(\sum_{i=1}^q h_{ii}^\alpha \right) \left(\text{tr} H_\alpha - \sum_{i=1}^q h_{ii}^\alpha \right) \right. \\ &\quad \left. - \frac{q}{n} \left(\sum_{k=q+1}^n h_{kk}^\alpha \right) \left(\text{tr} H_\alpha - \sum_{k=q+1}^n h_{kk}^\alpha \right) \right] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\alpha} \left[2 \sum_{k=q+1}^n \sum_{i=1}^q (h_{ik}^{\alpha})^2 - (n-q)T_{\alpha} \sum_{i=1}^q h_{ii}^{\alpha} + \frac{q(n-q)}{n} \sum_{i=1}^q (h_{ii}^{\alpha})^2 \right. \\
&\quad \left. - qT_{\alpha} \sum_{k=q+1}^n h_{kk}^{\alpha} + \frac{q(n-q)}{n} \sum_{k=q+1}^n (h_{kk}^{\alpha})^2 \right] \\
&\leq \frac{q(n-q)}{n} S - \sum_{\alpha} \left[qnT_{\alpha}^2 + (n-2q)T_{\alpha} \sum_{i=1}^q h_{ii}^{\alpha} \right] \\
&\leq q(n-q)[(n-1)(c+H^2) - Ric_{\min}] \\
&\quad - q(n-q)H^2 - (n-2q) \sum_{\alpha} \sum_{i=1}^q T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}). \tag{3.3}
\end{aligned}$$

It follows from (3.2), (3.3) and the assumption that

$$\begin{aligned}
&\sum_{k=q+1}^n \sum_{i=1}^q [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k) \rangle] \\
&\leq \frac{n-2q}{q(n-2)} \left\{ q(n-q)[(n-1)(c+H^2) - Ric_{\min}] \right. \\
&\quad \left. - q(n-q)H^2 + n(q-1) \sum_{\alpha} \sum_{i=1}^q T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}) \right\} \\
&\quad + \frac{n(q-1)}{q(n-2)} \left\{ q(n-q)[(n-1)(c+H^2) - Ric_{\min}] \right. \\
&\quad \left. - q(n-q)H^2 - (n-2q) \sum_{\alpha} \sum_{i=1}^q T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}) \right\} \\
&= q(n-q)[(n-1)(c+H^2) - Ric_{\min}] - q(n-q)H^2 \\
&< q(n-q)c. \tag{3.4}
\end{aligned}$$

This together with Theorem 2.1 implies that

$$H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0,$$

for all $2 \leq q \leq \frac{n}{2}$.

Since $(n-2)(c+H^2) \geq \frac{n(n-1)}{n+2}(c+H^2)$, we get from the assumption and Lemma 2.1 that

$$H_1(M; \mathbb{Z}) = H_{n-1}(M; \mathbb{Z}) = 0,$$

and M is simply connected.

From above discussion, we know that M is a homotopy sphere. This together with the generalized Poincaré conjecture implies that M is a topological sphere. This completes the proof of Theorem 3.1.

Theorem 3.1 improves the sphere theorem due to Vlachos [23] and Hu-Zhai [14]. The following example shows that our pinching condition in Theorem 3.1 is sharp.

Example 3.1. (i) Let $M := \mathbb{C}P^2(\frac{4}{3}(c + H^2)) \subset S^7(\frac{1}{\sqrt{c+H^2}})$, where H is a nonnegative constant. Then M is a compact minimal submanifold in $S^7(\frac{1}{\sqrt{c+H^2}})(\subset F^{4+p}(c))$ with $Ric_M \equiv 2(c + H^2)$. Hence M is a compact submanifold in $F^{4+p}(c)$ with constant mean curvature H . It is not a topological sphere.

(ii) Let $M := S^m(\frac{1}{\sqrt{2(c+H^2)}}) \times S^m(\frac{1}{\sqrt{2(c+H^2)}}) \subset S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m \geq 4$, where H is a nonnegative constant. Then M is a compact minimal submanifold in $S^{n+1}(\frac{1}{\sqrt{c+H^2}})(\subset F^{n+p}(c))$ with $Ric_M \equiv (n - 2)(c + H^2)$. Hence M is a compact submanifold in $F^{n+p}(c)$ with constant mean curvature H . It is not a topological sphere.

In the next, we investigate differentiable pinching problem on compact submanifolds in a Riemannian manifold, and obtain the following theorem.

Theorem 3.2. *Let (M, g_0) be an $n(\geq 4)$ -dimensional compact submanifold in an $(n + p)$ -dimensional Riemannian manifold N^{n+p} . If the Ricci curvature of M satisfies*

$$Ric_M > \left[\frac{3n^2 - 9n + 8}{3(n - 2)} \overline{K}_{\max} - \frac{8}{3(n - 2)} \overline{K}_{\min} \right] + \frac{n(n - 3)}{n - 2} H^2,$$

then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)} g(t),$$

exists for all time and converges to a constant curvature metric as $t \rightarrow \infty$. Moreover, M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n .

Proof. Set $T_\alpha = \frac{1}{n} tr H_\alpha$. Then $\sum_\alpha T_\alpha^2 = H^2$, and

$$\begin{aligned} h_{ii}^\alpha h_{jj}^\alpha &= \frac{1}{2} [(h_{ii}^\alpha + h_{jj}^\alpha - 2T_\alpha)^2 - (h_{ii}^\alpha - T_\alpha)^2 - (h_{jj}^\alpha - T_\alpha)^2] \\ &\quad + T_\alpha (h_{ii}^\alpha - T_\alpha) + T_\alpha (h_{jj}^\alpha - T_\alpha) + T_\alpha^2. \end{aligned} \quad (3.5)$$

We rewrite (2.3) as

$$\begin{aligned} Ric(e_i) &= \sum_j \overline{R}_{ijij} + (n - 1)H^2 + (n - 2) \sum_\alpha T_\alpha (h_{ii}^\alpha - T_\alpha) \\ &\quad - \sum_\alpha (h_{ii}^\alpha - T_\alpha)^2 - \sum_{\alpha, j \neq i} (h_{ij}^\alpha)^2. \end{aligned} \quad (3.6)$$

This implies that

$$\begin{aligned} - \sum_\alpha (h_{ii}^\alpha - T_\alpha)^2 &\geq Ric_{\min} - (n - 1)(\overline{K}_{\max} + H^2) \\ &\quad - (n - 2) \sum_\alpha T_\alpha (h_{ii}^\alpha - T_\alpha) + \sum_{\alpha, j \neq i} (h_{ij}^\alpha)^2, \end{aligned} \quad (3.7)$$

and

$$\sum_\alpha T_\alpha (h_{ii}^\alpha - T_\alpha) \geq \frac{1}{n - 2} [Ric_{\min} - (n - 1)(\overline{K}_{\max} + H^2)]. \quad (3.8)$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in \mathbb{R}$.

From (2.1), (2.4), (3.5), (3.7) and (3.8), we have

$$\begin{aligned}
& R_{1313} + R_{2323} - |R_{1234}| \\
= & \bar{R}_{1313} + \bar{R}_{2323} + \sum_{\alpha} \left[h_{11}^{\alpha} h_{33}^{\alpha} - (h_{13}^{\alpha})^2 + h_{22}^{\alpha} h_{33}^{\alpha} - (h_{23}^{\alpha})^2 \right] \\
& - |\bar{R}_{1234} + \sum_{\alpha} (h_{13}^{\alpha} h_{24}^{\alpha} - h_{14}^{\alpha} h_{23}^{\alpha})| \\
\geq & 2\bar{K}_{\min} - \frac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min}) - \frac{1}{2} \sum_{\alpha} \left[3(h_{13}^{\alpha})^2 + 3(h_{23}^{\alpha})^2 + (h_{14}^{\alpha})^2 + (h_{24}^{\alpha})^2 \right] \\
& + \sum_{\alpha} \left[-\frac{(h_{11}^{\alpha} - T_{\alpha})^2}{2} - \frac{(h_{33}^{\alpha} - T_{\alpha})^2}{2} + T_{\alpha}(h_{11}^{\alpha} - T_{\alpha}) + T_{\alpha}(h_{33}^{\alpha} - T_{\alpha}) + T_{\alpha}^2 \right] \\
& + \sum_{\alpha} \left[-\frac{(h_{22}^{\alpha} - T_{\alpha})^2}{2} - \frac{(h_{33}^{\alpha} - T_{\alpha})^2}{2} + T_{\alpha}(h_{22}^{\alpha} - T_{\alpha}) + T_{\alpha}(h_{33}^{\alpha} - T_{\alpha}) + T_{\alpha}^2 \right] \\
\geq & \frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) - \frac{1}{2} \sum_{\alpha} \left[3(h_{13}^{\alpha})^2 + 3(h_{23}^{\alpha})^2 + (h_{14}^{\alpha})^2 + (h_{24}^{\alpha})^2 \right] \\
& + 2[Ric_{\min} - (n-1)(\bar{K}_{\max} + H^2)] + 2H^2 \\
& + \frac{1}{2} \sum_{\alpha, j \neq 1} (h_{1j}^{\alpha})^2 + \frac{1}{2} \sum_{\alpha, j \neq 2} (h_{2j}^{\alpha})^2 + \sum_{\alpha, j \neq 3} (h_{3j}^{\alpha})^2 \\
& + \frac{n-4}{2} \sum_{\alpha, i \neq 1, 3} T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}) + \frac{n-4}{2} \sum_{\alpha, i \neq 2, 3} T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}) \\
\geq & \frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) + 2H^2 + (n-2)[Ric_{\min} - (n-1)(\bar{K}_{\max} + H^2)]. \tag{3.9}
\end{aligned}$$

Same argument implies that

$$\begin{aligned}
& R_{1414} + R_{2424} - |R_{1234}| \\
\geq & \frac{8}{3} \left(\bar{K}_{\min} - \frac{1}{4} \bar{K}_{\max} \right) + 2H^2 + (n-2)[Ric_{\min} - (n-1)(\bar{K}_{\max} + H^2)]. \tag{3.10}
\end{aligned}$$

This together with (3.9) and the assumption implies

$$\begin{aligned}
& R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
\geq & R_{1313} + R_{2323} - |R_{1234}| + \lambda^2 (R_{1414} + R_{2424} - |R_{1234}|) \\
> & 0. \tag{3.11}
\end{aligned}$$

It follows from Theorem A that M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n . This completes the proof of Theorem 3.2.

Theorem 3.3. *Let M be an $n(\geq 4)$ -dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. If*

$$Ric_M > (n-2)(1 + \varepsilon_n)(c + H^2),$$

then M is diffeomorphic to S^n . Here

$$\varepsilon_n = \begin{cases} 0, & \text{for } 4 \leq n \leq 6, \\ \frac{n-4}{(n-2)^2}, & \text{for } n \geq 7. \end{cases}$$

Proof. When $n = 5, 6$, it is well known that there is only one differentiable structure on S^n . This together with Theorem 3.1 implies M is diffeomorphic to S^n . When $n \neq 5, 6$, it follows from Theorem 3.2 that M is diffeomorphic to a spherical space form. On the other hand, it follows from Lemma 2.1 that M is simply connected. Therefore, M is diffeomorphic to S^n . This completes the proof of Theorem 3.3.

Remark 3.1. When $4 \leq n \leq 6$, the pinching condition in Theorem 3.3 is sharp. When $n \geq 7$, we have $0 \leq \varepsilon_n < \frac{1}{n}$ and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Therefore, the pinching condition in Theorem 3.3 is close to the best possible.

Motivated by Theorem E and the sphere theorems above, we would like to propose the following conjecture.

Conjecture. Let M be an $n(\geq 3)$ -dimensional compact oriented submanifold in the space form $F^{n+p}(c)$ with $c + H^2 > 0$. If

$$\text{Ric}_M \geq (n-2)(c + H^2),$$

then M is diffeomorphic to either the standard n -sphere S^n , the Clifford hypersurface $S^m(\frac{1}{\sqrt{2}}) \times S^m(\frac{1}{\sqrt{2}})$ in S^{n+1} with $n = 2m$, or $\mathbb{C}P^2$. In particular, if $\text{Ric}_M > (n-2)(c + H^2)$, then M is diffeomorphic to S^n .

Theorems 3.2 and 3.3 provide partial affirmative answer to the Conjecture.

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